How Electronic Voting Can Escape Arrow's Impossibility Theorem

Jean-Luc Koning

Institut National Polytechnique de Grenoble, France
Jean-Luc.Koning@esisar.inpg.fr,
WWW home page: http://www.esisar.inpg.fr/lcis/cosy

Abstract. This paper discusses possible voting procedures as they are described in classical collective choice theory and shows some of their limits. No classical voting procedure can be at the same time democratic, decisive and rational. Because of this result, seeking a way to aggregate electronic votes stemming from various individuals that satisfy these three properties is vain when candidate rankings alone are taken into account.
To overcome these limits, this paper introduces preference based voting procedures, states the conditions such an aggregation function should satisfy (much in the same way as Arrow did but in a fuzzy setting), and explains how voters could naturally derive degrees of preference among candidates so as to lead to feasible electronic voting procedures.

Keywords: electronic voting, social choice theory, arrow's theorem.

1 Introduction

Social choice theory, elaborated by economists and social scientists [1], aims at studying voting procedures, i.e., where voters have to choose among candidates. Each voter ranks the candidates according to his/her preferences. In order to obtain a global ranking of the candidates, the individual rankings are aggregated. The way to aggregate the rankings is devised according to the properties one may want the voting procedure to satisfy.

This paper discusses possible voting procedures as they are described in classical collective choice theory and shows some of their limits. No classical voting procedure can be at the same time democratic, decisive and rational. Because of this result, seeking a way to aggregate electronic votes stemming from various individuals that satisfy these three properties is vain when candidate rankings alone are taken into account.

To overcome these limits, this paper introduces preference based voting procedures, states the conditions such an aggregation function should satisfy (much in the same way as Arrow did but in a fuzzy setting), and explains how voters could naturally derive degrees of preference among candidates so as to lead to feasible electronic voting procedures.

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1.1 Some Classical Results

Terminology

Arrow has proposed conditions every voting procedure[1] should satisfy [5]. He also proved a very important theorem. Before recalling it, let us specify the terminology used in the following.

An individual vote is the ranking of all the candidates by decreasing order of preference for the elector. It is not the exclusive expression of the preferred candidate. The notion of individual corresponds to what is called sometimes *weak preference*. One writes $x \succsim y$ to mean that an individual prefers candidate x at least as much as candidate y. This weak preference relation can be split into two sub-relations: $x \succ y$ means candidate x is strictly preferred to candidate y ($y \not\succsim x$), and $x \sim y$ means the voter is indifferent between the two alternatives ($x \succsim y$ and $y \succsim x$). By definition weak preference is a complete asymmetric pre-order.

Arrow's Impossibility Theorem

A voting procedure is a function that provides a collective ranking from the individual rankings given by the voters. For instance, (1) the *plurality* rule takes into account the number of times a candidate is at the top of any individual ranking. The greater this number, the higher the candidate is in the global ranking. (2) The *intensity* rule gives a candidate as many tokens as the number of candidates he/she outranks in an individual ranking. The tokens a candidate receives for each individual ranking are added and this total score determines the position of the candidate in the global ranking. (3) In the *majority* rule, candidate x is preferred over another candidate y if and only if the number of individuals ranking x before y is at least as large as the number of individuals ranking y before x. In the case of a panel of two candidates these voting procedures are equivalent.

For three or more alternatives possible ambiguities arise; i.e., none of these voting procedure is completely satisfactory.

Properties of a Voting Procedure

Arrow tackled this issue by stating the properties a voting procedure should satisfy when there are at least three candidates x, y and z:

Completeness The voting procedure ranks each candidate pairwise (i.e., it does not exist two candidates for whom one does not know the weakly preferred one): $x \succsim y$ or $y \succsim x$.

Transitivity In the final collective ranking the preference relation must be transitive: if $x \succsim y$ and $y \succsim z$ then $x \succsim z$.

[1] The term *voting procedure* is widely used in everyday life. K. Arrow calls them *social welfare functions* [2]. Some call them *group decision functions* [3] or *collective choice rules* [4], because Arrow's terminology may be mistaken with a function measuring welfare.

Unrestricted domain The voting procedure is defined whatever the individual votes are.

Unanimity If each individual prefers candidate x to candidate Y, the collective ranking must then prefer x to y.

Independence of irrelevant alternatives The collective ranking of any pair of candidate only depends upon individual rankings of the candidates from this pair. For instance, if one is to find out whether a group of people prefers coffee or tea, individual preferences between tea and coke must not influence this choice.

1.2 Ideal Voting Procedure

After having defined and justified these properties, Arrow showed in the theorem bearing his name, that the only voting procedure that verifies them is the dictatorship (i.e., one individual imposes his choice on the others) in the case of three or more candidates.

In electronic voting, because of this result, seeking a voting procedure that satisfy these five properties is vain. It will consist in selecting a particular vote and rejecting the others if one *only* takes into account the ranking induced on the candidates. So as to avoid the conclusion of Arrow's theorem, one may want either to abandon or weaken one or several properties and find a voting procedure compatible with this new set of axioms. Unfortunately, relaxing the initial properties make the voting procedures counterintuitive or little decisive (e.g., they allow preference intransitivity in the collective ordering), as long as they are not anti-democratic. The ideal voting procedure thus does not exist.

The right of veto of an individual is defined as the possibility to object to a strict collective preference that is different from his/her own. A voting procedure that would allocate a right of veto to each voter in order to distribute in an egalitarian way the power, would rarely be decisive since two opposite preferences on a pair of candidates result in indifference or conflict. An oligarchy is a set of voters that has, as a group, the power to impose on the remaining voters, its strict and unanimous preference for any pair of candidate. Moreover, each oligarchy member, as an individual, has the power to impose its veto against a strict collective preference different from his own. When an oligarchy contains only one individual it is a dictatorship. The more members it contains, the more egalitarian the power distribution. However, an increase in the size of an oligarchy increases indecisiveness at the same time.

2 Aggregating Electronic Votes

Let us analyze the problem of aggregating electronic votes. An elector's vote is a list of ranked candidates. Each elector may have an opinion on a candidate that differs from the others or even have no opinion at all on certain candidates. Consequently, candidate rankings given by voters do not necessarily contain the very same candidates even though some of them may be encountered in several lists.

Preference Intensity Vectors rather than mere Partial Pre-Orders
Arrow's impossibility theorem proves that it is not sufficient to bring into play rankings in order to obtain satisfactory preference aggregation modes (i.e., different from dictatorship and nevertheless meeting the five properties stated in Section 1.1). The expressive power of preference intensity vectors being greater than mere partial pre-orders, one may hope to somehow escape the theorem's conclusions by adding to the preference rankings an intensity measure.

This approach meets Yager's [6] for whom aggregating a group of preferences becomes possible when a ranking scale, more meaningful than a mere pre-order, is introduced. He also expresses preferences by means of fuzzy sets that convey to what extent candidates satisfy a voter. This enables a vote to be represented by a ranked and valued list of candidates. Those valuations can be obtained owing to the gradual —fuzzy— character of the elector.

Votes as Discrete Fuzzy Sets
Let $\Delta = \{\delta_1, \ldots, \delta_m\}$ be a finite non-empty set of alternatives that are to be evaluated by a board of experts $\mathcal{A} = \{A_1, \ldots, A_n\}$.

Each expert is supposed to rank every alternative according to a continuous scale that is, by convention, the unit interval $[1, 0]$.

Let x_{ij} be the rating for alternative δ_i given by expert A_j. For instance Δ a set of artists the board of examiners $\mathcal{A}$ has to rank according to their skills, by using (as often) a numerical grading scale. Each expert comes up with a ranking F_j that may be considered as a fuzzy set whose membership function μ_{F_j} such that $x_{ij} = \mu_{F_j}(\delta_i)$. F_j is the discrete fuzzy set of alternatives preferred by expert A_j, x_{ij} may be seen as the matching degree between δ_i and some ideal alternative according to A_j.

In a vote, candidates are ranked by decreasing order of degree (x_{ij}) to which the voters advocate them. This coefficient grades the interest for a candidate δ_i by a voter A_j. By definition $x_{ij} = 1$ conveys maximal preference, $x_{ij} =$ rejection and $x_{ij} = 0.5$ indifference about candidate δ_i. With this definition the vote of an elector A_j can be represented by a discrete fuzzy set F_j on Δ. Figure shows an example of such a preference ranking.

Once the candidate rankings are replaced by fuzzy sets as just defined, it interesting to carry Arrow's properties (among others) over to this formalism order to find the fuzzy voting procedures that satisfy them.

This study has been carried out in a systematic manner on Arrow's properties and also on different conditions which a voting procedure may satisfy.

2.1 Axioms

The problem of electors who vote then becomes a problem of finding a function for aggregating n fuzzy subsets on the set Δ of possible candidates into a fuzzy set F. It is thus a matter of finding f such that $f(F_1, F_2, \ldots, F_n) = F$. Let see some of the conditions an aggregation function f should satisfy.

All possible axioms are not required with the same strength, and do not pertain to the same purpose. They can be classified into three groups:

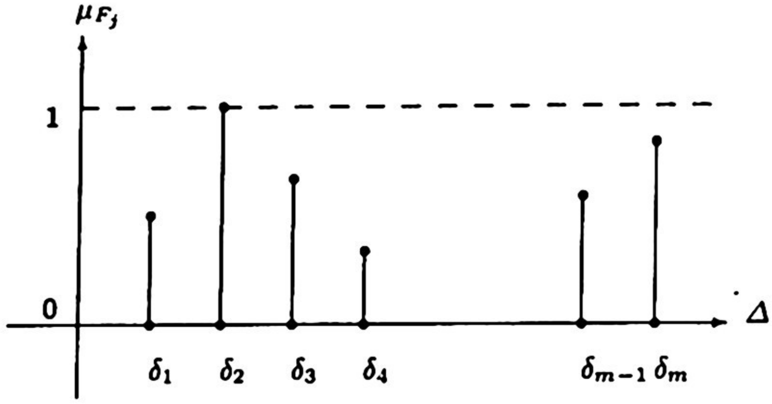

Fig. 1. Discrete fuzzy set of candidates advocated by voter A_j

1. *Imperative conditions* whose violation lead to obviously counterintuitive aggregation modes.
2. *Technical conditions* that just facilitate the representation on the calculation of the aggregation function.
3. *Optional conditions* that naturally apply in special circumstances but are not necessarily universally accepted. They are acceptable as long as they do not lead to impossibility results, or restrict too much the range of admissible aggregations.

Imperative Conditions

Insofar as a democratic aggregation is assumed, the imperative conditions are:

Unanimity in rejection, acceptance and indifference: If every voter is indifferent with respect to a candidate then the final candidate ranking follows this general consensus ($\exists \delta_i$, $\forall j$, $x_{ij} = .5 \Longrightarrow \mu_F(\delta_i) = .5$), the same is true for acceptance ($x_{ij} = 1 \Longrightarrow \mu_F(\delta_i) = 1$) and rejection ($x_{ij} = 0 \Longrightarrow \mu_F(\delta_i) = 0$).

Positive association of social and individual values (***in its non-strict form***): If an individual increases his preference intensity for δ_i then the social preference for δ_i cannot decrease. It means that if F'_j and F_j are such that $\mu_{F_j} \leq \mu_{F'_j}$ (i.e., in fuzzy set terms, $F_j \subseteq F'_j$) then $f(F_1 \ldots F_j \ldots F_n) \subseteq f(F_1 \ldots F_j \ldots F_n)$. In other words f is monotonic with set-inclusion.

Minimal democracy: No one has absolute veto nor is a dictator in every situation.

Neutrality with respect to alternatives: The aggregation function does not depend upon the alternatives. If δ_i and $\delta_{i'}$ are such that $x_{ij} = x_{i'j}, \forall j \in \{1, \ldots, n\}$ then $\mu_{f(F_1 \ldots F_n)}(\delta_i) = \mu_{f(F_1 \ldots F_n)}(\delta_{i'})$.

Neutrality with respect to the intensity scale: Assume that the alternatives are rated in terms of *distaste* intensities instead of preference intensities. Then the social distaste pattern should be built from individual distaste function with the same aggregation function as preferences. Indeed distaste and preference are just a matter of naming the assessment criterion (i.e., choosing the *good* or the *bad* alternatives) and the aggregation function should not depend on this name.

Continuity An infinitesimal modification in the individual preference intensities only induces an infinitesimal modification in the collective preferences.

Technical Condition

One technical condition would be the *independence of irrelevant alternatives* (see Arrow's fifth condition in Section 1.1) which requests that the social preference intensity for δ_i only depends on the individual preference intensity for δ_i, and not for $\delta_{i'}$, $i' \neq i$. Together with the neutrality of alternatives it enables the aggregation function to be represented by a mapping $\phi : [0,1]^n \longrightarrow [0,1]$ such that

$$\forall \delta_i,\ i = \{1,\ldots,m\},\ \mu_F(\delta_i) = \phi(x_{i1}, x_{i2}, \ldots, x_{in}) \stackrel{\text{def}}{=} x_i$$

2.2 Fuzzy Voting Procedures

If this last technical condition is added to the imperative ones then the qualified class of fuzzy set operators for voting-like aggregation procedures is the class of *symmetric sums* that has been fully studied in fuzzy set theory by Silvert [7] [8].

On the whole, the results investigated here can be compared to those that have been found following Arrow. Namely in classical voting theory, no voting procedure can be at the same time democratic, decisive and rational. Here, rational means "obeying Arrow's axioms". Moreover, three voting rules satisfy only two of these requirements:

- *simple majority* is democratic and decisive,
- *unanimity* is democratic and rational,
- *dictatorship* is decisive and rational.

In the fuzzy set setting the following families of aggregation operations are many-valued counterparts of these three rules. The *median* and the *minimum* operations match with the unanimity rule, the *arithmetic mean* matches with the majority rule, the *maximum operation* and the *associative symmetric sums* match with more or less tough dictatorships.

Associative Symmetric Sums

These functions admit the following properties:

- Negative individual opinions result in reinforced social negative opinion and the same for positive opinion, while contradictory opinion compensate. If $x_{ij} < 0.5 < x_{ik}$ then $\phi(x_{ij}, x_{ij}) < x_{ij}, \phi(x_{ik}, x_{ik}) > x_{ik}$ and $\phi(x_{ij}, x_{ik}) \in [x_{ij}, x_{ik}]$.

- Social choice is forbidden by the existence of extreme conflict between any two individuals in a society. If $\exists x_{ij}, x_{ik}$ such that $x_{ij} = 0, x_{ik} = 1. \forall k \neq j$ then $\phi(x_{i1}, \ldots, x_{in})$ is undefined. Arrow's third axiom (*unrestricted domain*) is thus violated.
- An indifferent vote does not modify the collective choice, $\phi(x_{ij}, x_{ik}, 0.5) = \phi(x_{ij}, x_{ik})$.
- Any individual has a right of veto against others when they do not oppose him/her completely. If $\exists x_{ij} = 0$ and $x_{ik} < 1, \forall k \neq j$ then $\phi(x_{i1}, \ldots, x_{in}) = 0$.
- Any individual can be a dictator if other individuals do not completely reject his/her choice. If $\exists x_{ij} = 1$ and $x_{ik} > 0, \forall k \neq j$ then $\phi(x_{i1}, \ldots, x_{in}) = 1$.

Associativity belongs to the set of optional conditions and is not always desirable because it leads to weighing the opinion of a group of $n-1$ individuals in the same way as the opinion of the nth individual. This state of fact is not acceptable when individual veto or dictatorship is rejected.

An example of such a function is:

$$\phi(x_{i1}, x_{i2}, \ldots, x_{in}) = \frac{x_{i1} \cdot x_{i2} \ldots x_{in}}{x_{i1} \cdot x_{i2} \ldots x_{in} + (1 - x_{i1}) \cdot (1 - x_{i2}) \ldots (1 - x_{in})}$$

This class of functions satisfy all of Arrow's properties (rationality, decisiveness and democracy).

Note: In the fuzzy set setting one can also derive aggregation operations that are many-valued counterparts of the three rules given in Section 1.1.

Means

These functions are the counterparts of the majority rule. They admit the following properties:

- *Impotency:* This is a strong unanimity condition that does not enable reinforcement effect in preference intensities. If $x_{i1} = x_{i2} = \ldots = x_{in} = x_i$ then $\phi(x_i, \ldots, x_i) = x_i$.
- Adding or withdrawing an indifferent vote modifies the global result by approaching or getting away from the collective indifference. If $x_{ij} < 0.5$ (resp. $x_{ij} > 0.5$) then $\phi(x_{ij}, 0.5) \in [x_{ij}, 0.5]$ (resp. $[0.5, x_{ij}]$).

The typical operation of this kind is the arithmetic mean:

$$\phi(x_{i1}, x_{i2}, \ldots, x_{in}) = \frac{x_{i1} + x_{i2} + \ldots + x_{in}}{n}$$

which is a well known utilitarian aggregation [1].

Note: The only associative symmetric sum that is a mean is the median:

$$\phi(x_{i1}, x_{i2}, \ldots, x_{in}) = med(min_j(x_{ij}), max_j(x_{ij}), 0.5)$$

but this function is not very decisive since it leads to indifference as soon as there is some contradiction in the group of voters, i.e., as soon as $\exists x_{ij} \leq 0.5, x_{ik} \geq 0.5$. This rule is the counterpart of the unanimity in social choice theory [9].

Minimum and Maximum

These operations respectively express the right of veto and the right of dictatorship for any individual. They are thus anti-democratic although they do respect the *minimal democracy* axiom.

With the minimum operation, collective acceptance is hard to reach but collective rejection is attained as soon as there exists an individual rejection. In other respects, the veto (resp. dictatorship) effect attached to the minimum (resp. maximum) can be softened by defining the collective rejection (resp. acceptance) as the rejection (resp. acceptance) by a certain number of individuals. This number can be given in an approximate way by means of fuzzy quantifies.

Note: The median and the minimum operations match with the unanimity rule, the arithmetic mean matches with the majority rule, the maximum operation match with dictatorships. Thus, in the context of the voters previously identified, one can attach a —possibly weighted— maximum operation to the class of dictators, either an arithmetic mean or an associative symmetric sum to the class of democratic agents (depending on the role one may want indifference —preference degree of .5— to play), and a —possibly weighted— minimum operation to the class of veto agents.

3 Summary

No classical voting procedure can be at the same time democratic, decisive and rational. This is mainly due to the fact that only candidate rankings are taken into account. It turns out that mere partial pre-orders do not have enough expressive power. On the other hand, by using preference intensity vectors, votes can be conveyed by means of discrete fuzzy sets. Therefore, it becomes possible to implement electronic voting procedures that escape Arrow's impossibility theorem. This amounts to determining fuzzy aggregation functions that follow the set of Arrow's (fuzzy) conditions.

The class of functions that satisfy Arrow's general properties (rationality, decisiveness and democracy) is the set of associative symetric sums that are easy to implement in an electronic voting setting.

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